IES San Felipe Neri
Departamento de Matemáticas
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UNIT 7: FUNCTIONS.

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3º ESO Mathematics for academic studies

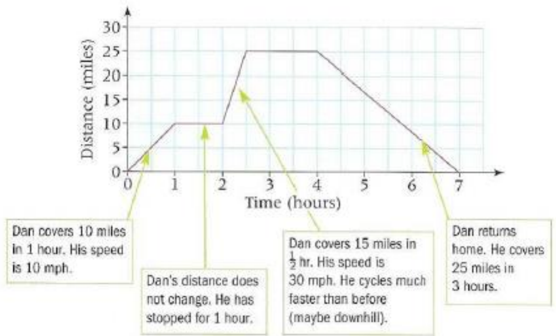
1. Functions and graphs

In everyday life, many quantities depend on one or more changing variables. For example:

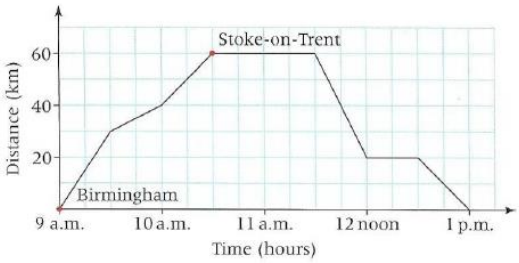
- Plant growth depends on sunlight and rainfall.
- Distance travelled depends on speed and time taken.

Graphs can be used to quickly get an idea of how one quantity varies as another quantity changes. Let's see an example:

This graph shows Dan's journey on his bike:



Exercise 1. The distance-time graph show the journey of a car between Birmingham and Stoke-on-Trent:



a) How far is it from Birmingham to Stoke-on-Trent?
b) For how long did the car stop?
c) What has the speed of the car for the first part of the journey?
d) Between which two times was the car travelling fastest?
e) What was the average speed of the car for the whole journey?

a. Function

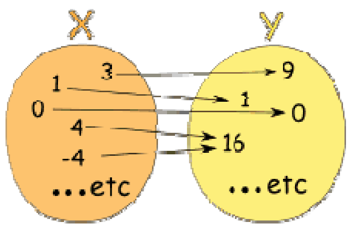
A **function** is a relation between two variables such that for every value of the first, there is only **one** corresponding value of the second. We say that the second variable is a function of the first variable.

The first variable is the **independent variable** (usually x) and the second variable is the **dependent variable** (usually y). **NOTE: y=f(x)**

Examples:

1) The formula of the area of a circle is $A = \pi r^2$. This is a **function** as each value of the independent variable **r** gives you one value of the dependent variable A.

2) In the equation $y = x^2$, **y** is a function of **x**, since for each value of x there is only one value of y.



3) Suppose a notebook costs €1.50 and we want to write the relationship of dependence between the number of notebooks we buy and the total cost that we pay for:
x = number of notebook we buy
y = f(x) = total cost we pay for

A function can be described in some different ways:

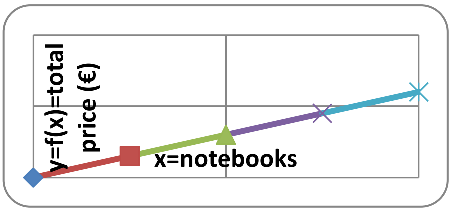
a) Using a formula: $f(x) = 1.5x$

b) Using a table:

x	1	2	3	4		
y			1.5	3	4.5	6

It means that when x=2, then y=3 (If you buy 2 notebooks, you pay €3.00), and so on.

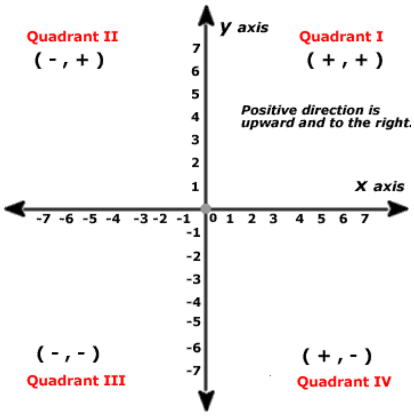
c) Using a graph: It means drawing the function by plotting the pair of coordinates we obtained making the table:
The points obtained are: (1, 1.5), (2, 3), (3, 4.5), (4, 6)



By studying the graph you see that the more notebooks you buy, the more money you pay; and you are able to see the values of y for each given value of x.

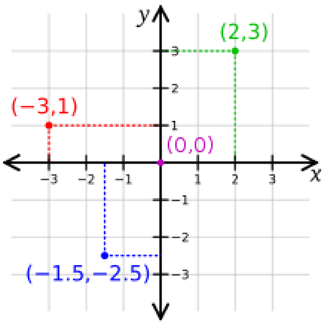
b. Graphic representation

i. Cartesian plane, abscissa axis, ordinate axis.

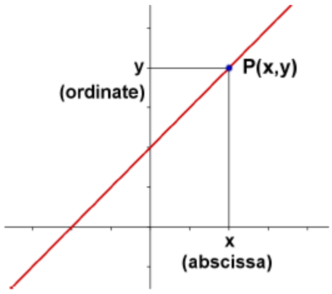


The French philosopher, mathematician and scientist **René Descartes** (17th century) was one of the most important and influential thinkers in history. Descartes created the coordinate system we are going to study, which is also called the **Cartesian plane**.

The coordinate system is formed by two lines: A horizontal line that is called an **x-axis** and a vertical line that is called a **y-axis**. Both lines meet at a point called an **origin**. The distances from the origin towards either the right or upwards are positive, and the distances from the origin towards either the left or downwards are negatives.



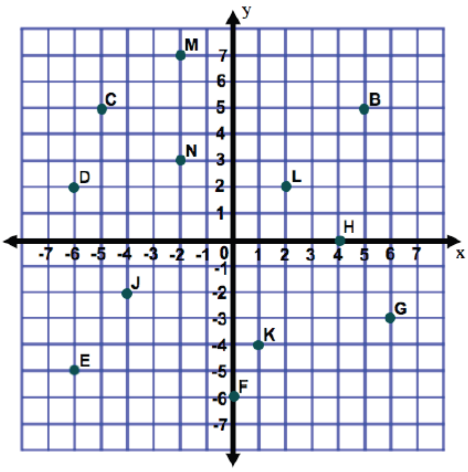
The two axes divide the plane into four regions or **quadrants**.
The location of any point on the coordinate plane is denoted by using a pair of coordinates (x, y), where X is the horizontal distance from the origin and Y is the vertical distance from the origin.



The x-value, called the **abscissa**, is the perpendicular distance of P from the y-axis.

The y-value, called the **ordinate**, is the perpendicular distance of P from the x-axis.

The values of x and y together, written as (x,y) are called the **coordinates** of the point P.



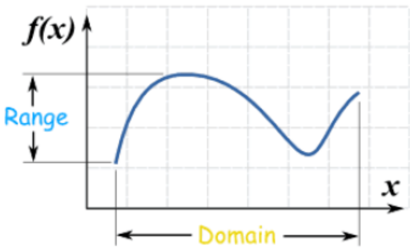
Exercise 2. Name the point that has the coordinates.
a. (2, 2) b(-6, 2) c.(1,-4) d.(0,-6) e (-4,-2)

Exercise 3. Write the coordinates of each point.
a. B b. G c. E d. N e. H

Exercise 4. In what quadrant is each point located?
a. C b. J c. L d. M e. K

2. Characteristics of a function

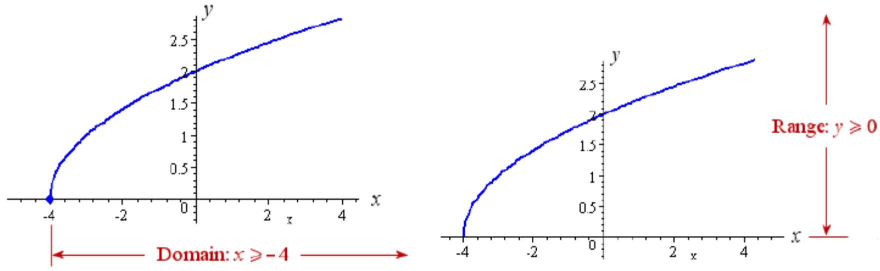
a. Domain and Range



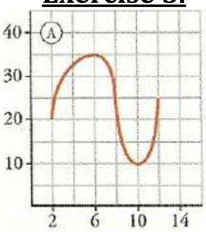
The **domain** of a function is the complete set of possible values of the independent variable in the function.

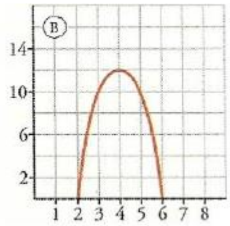
The **range** of a function is the complete set of all possible resulting values of the dependent variable of a function, after we have substituted the values in the domain.

Example: Find the domain and the range for the function $y=\sqrt{x+4}$.



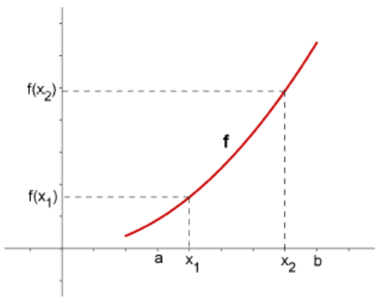
Exercise 5. Find the domain and the range for the following functions:

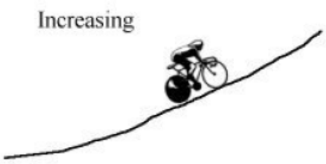


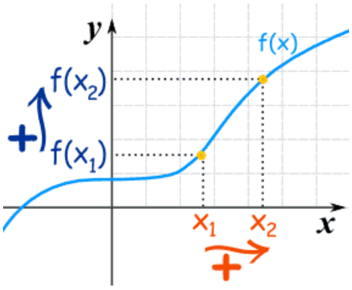


b. Increase and decrease of a function

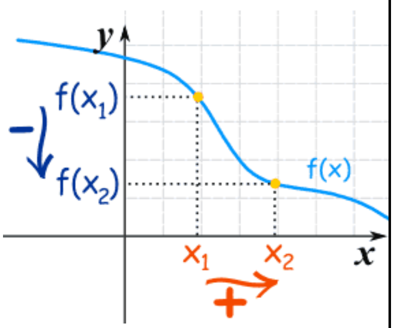
A function f is **increasing** on an interval (a, b) if for any x_1 and x_2 in the interval such that $x_1 < x_2$ then $f(x_1) < f(x_2)$. Another way to look at this is: as you trace the graph from a to b (that is from left to right) the graph should go up.

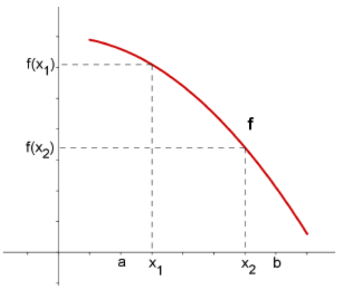


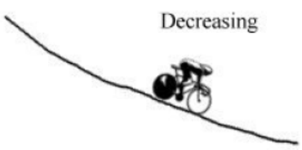




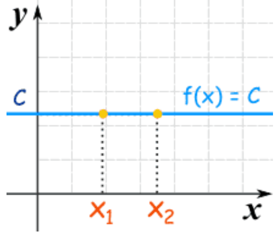
A function f is **decreasing** on an interval (a, b) if for any x_1 and x_2 in the interval such that $x_1 < x_2$ then $f(x_1) > f(x_2)$. Another way to look at this is: as you trace the graph from a to b (that is from left to right) the graph should go down.



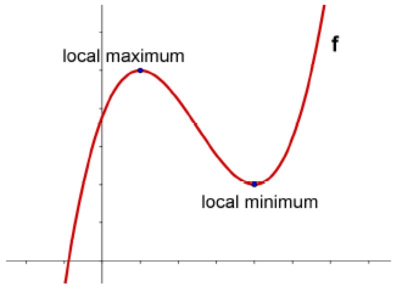




A **Constant Function** is a horizontal line:

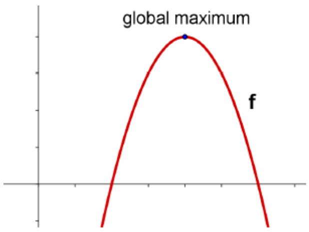


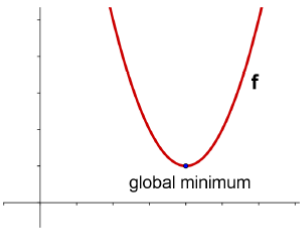
c. Relative minimums and maximums



A function f has a **relative (or local) maximum** at a point if its ordinate is greater than the ordinates of the points around it.
A function f has a **relative (or local) minimum** at a point if its ordinate is less than the ordinates of the points around it.

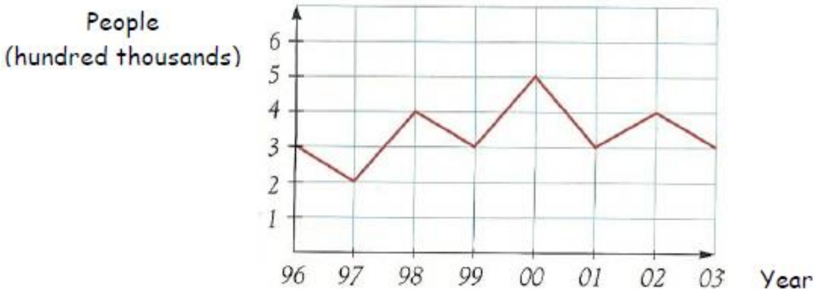
d. Absolute minimums and maximums



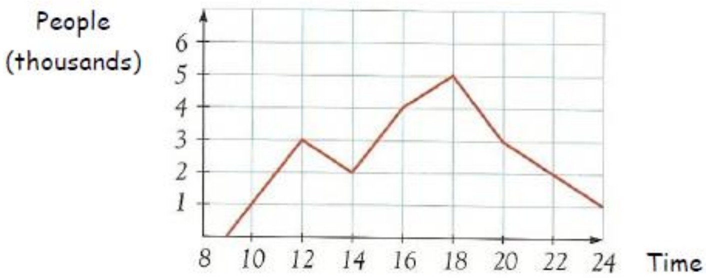


A function f has an **absolute (or global) maximum** at a point if its ordinate is the largest value that the function takes on the domain that we are working on.
A function f has an **absolute (or global) minimum** at a point if its ordinate is the smallest value that the function takes on the domain that we are working on.

<u>Exercise 6.</u> The following graph shows the number of people who have lived in a city during some years. Determine the intervals of increasing and decreasing of this function as well as its relative and absolute maximums and minimums.



<u>Exercise 7.</u> The following graph shows the present people (in thousands) in a shopping centre during a day. Determine the intervals of increasing and decreasing of this function as well as its relative and absolute maximums and minimums.



<u>Exercise 8.</u> Draw the graph of a function with the following features:

- Local maximum at x=-2 and x=3.
- Local minimum at x=1 and x=2.
- A global maximum at x=0.
- A global minimum at x=-1.

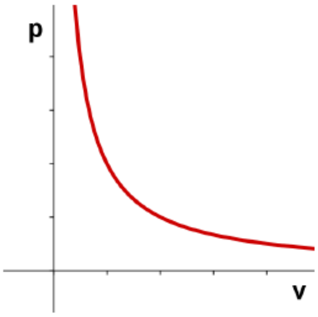
e. Trend of a function

i. Long-term behaviour

Do you remember what the relationship between pressure and volume is?

For a fixed amount of any gas, kept a fixed temperature, pressure and volume are linked by the formula $p = \frac{k}{v}$, where p is the pressure, v is the volume and k is a constant. A gas cannot have a negative volume, so the values of v will be greater than 0.

The graph for this formula is shown opposite:



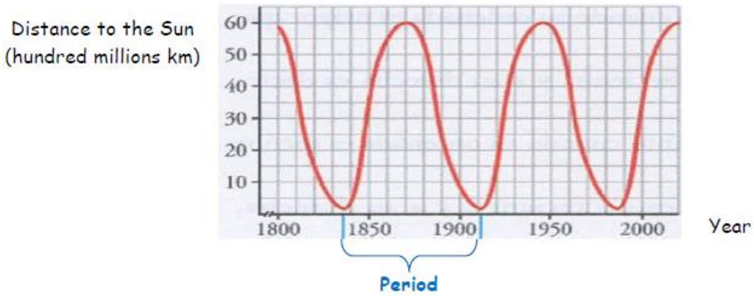
The pressure of the gas gets higher as the volume gets smaller.

As the volume increases, the pressure gets close to zero. As the volume gets closer to zero, the pressure gets higher, but the volume never actually equals zero.

There are functions where, although you only know a piece of them, you can guess how they will work far away from that piece. As you have seen in the previous example, these functions have "branches" with a very clear tendency.

ii. Periodicity

The following graph shows the distance from the Halley's Comet to the Sun during the last two centuries.

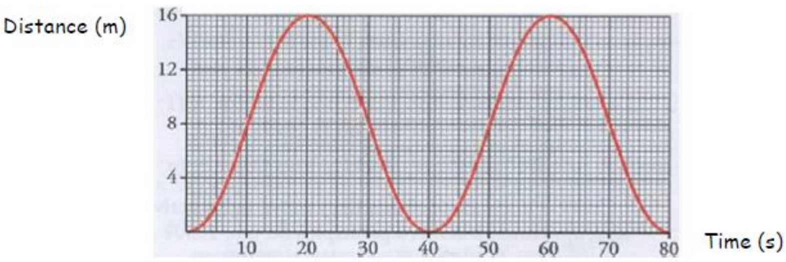


Since the orbit is repeated over and over every 76 years, the graph is also repeated in this period of time. This is a periodic function with period 76.

A **periodic function** is a function which has a graph that repeats itself identically over and over as it is followed from left to right.

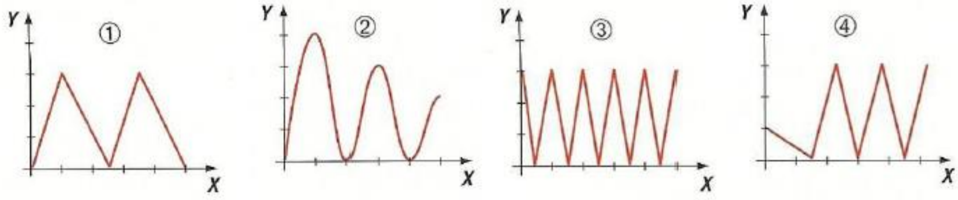
The horizontal distance required for the graph of a periodic function to complete one cycle is called **period**.

Exercise 9. When a big wheel turns, its baskets go up and go down. This is the graphical representation of the function time-distance to the floor of one of these baskets.



a) How long does the basket take to complete a turn?
b) What is the maximum height? What is the radius of the big wheel?
c) Explain how to calculate the height at 130 seconds without going on the graph.

Exercise 10. Which of the following functions are periodic? What are their periods?

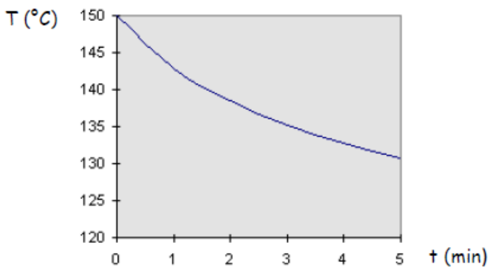


f. Continuity and discontinuity

Example 1: Steam in a boiler was heated to 150°C. Its temperature was then recorded each minute as follows:

Time (min)	0	1	2	3	4	5
Temp (°C)	150.0	142.8	138.5	135.2	132.7	130.8

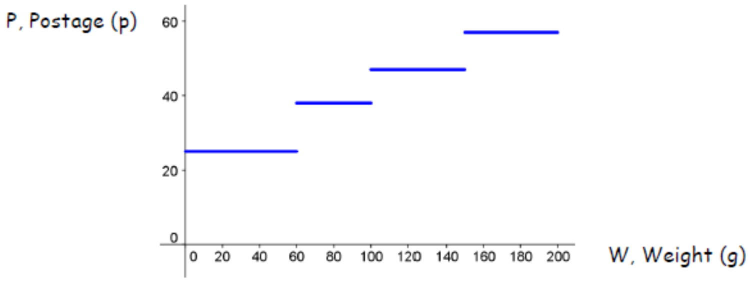
Since the temperature changes in a continuous way, there is meaning to the values in the intervals between the points. Therefore, the points are joined by a smooth curve. This is a continuous function.



The graph of a **continuous function** can be drawn without lifting the pencil from the paper.

Example 2: The table below shows the postal rates in the U.K. for first class letters in 2006.

Weight up to	60 g	100 g	150 g	200 g
Postage	25 p	38 p	47 p	57 p



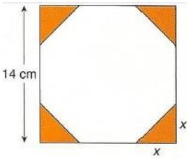
The "jumps" of this graph are called **discontinuities** of the function.

Graphs of **discontinuous functions** cannot be drawn without lifting the pencil from the paper.

A function is <u>continuous</u> when it has no discontinuities. In other words, you can draw a graph of it without lifting your pencil from the paper.
You can also say that a function is continuous in an interval, even if it has discontinuities in other places.

Exercise 11. The weight of one pound is 0,45kg. Obtain the analytical expression of the function that converts pounds in kg. Draw its graph.

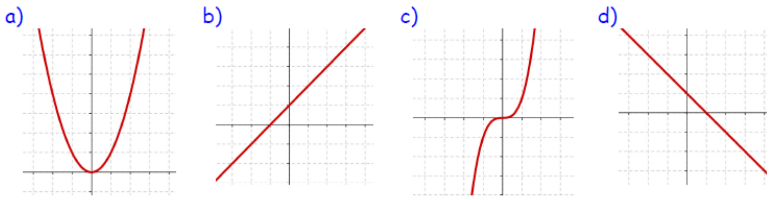
Exercise 12. An architect designs a window such that it has the shape of a rectangle with a semicircle on top, as shown. The architect wants the base of the window to be 10 cm less than the height of the rectangle part. Express the perimeter P of the window as a function of the radius r of the circular part.



Exercise 13. The side of a square is 14 cm. Four isosceles right triangles are cut out from the corners. Express the area of the remainder octagon as a function of x.

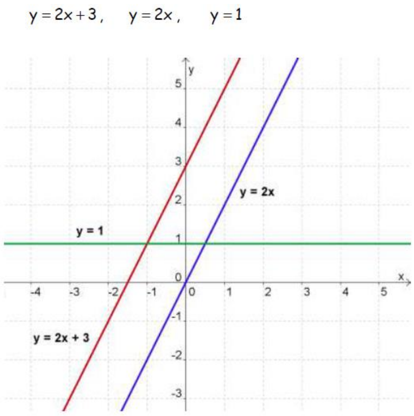
Exercise 14. Match each graph with one of the following analytical expressions:

1) $y = x + 1$ 2) $y = x^3$ 3) $y = x^2$ 4) $y = -x + 1$



3. Linear functions. Straight line. y=mx+n

A linear function of a single variable has the form $f(x) = mx + n$.
The graph for this function is a straight line with **slope** *m.*



a. Representing the equation of a straight line

Make table of values with three values.

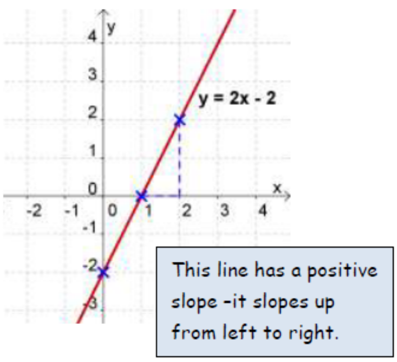
b. Slope and ordinate at the origin

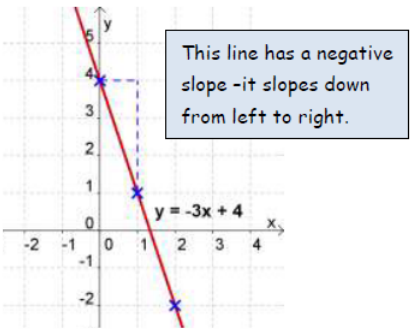
- **Slope (m):** The slope is the coefficient of x in the equation y=mx+n. It represents the variation in y for each unit of x.
- **Ordinate at the origin (n):** If x=0, then y=n. So, it intersects the Y axis at point (0,n).

- Straight lines with **positive slope** are **increasing functions.**
- Straight lines with **negative slope** are **decreasing functions.**
- The **slope** of **constant functions** is **0.** Their graphs are horizontal lines.

The **slope (or gradient)** of a straight line tells you how steep it is.

To work out the slope find how many units the line **rises** for each unit it **runs** across the page.





For the line $y = 2x - 2$,

$$\text{slope} = \frac{\text{rise}}{\text{run}} = \frac{2}{1} = 2$$

For the line $y = -3x + 4$,

$$\text{slope} = \frac{\text{rise}}{\text{run}} = \frac{-3}{1} = -3$$

c. Function of proportionality y=mx

Linear functions in which n=0, that is, **y=mx**, are called **proportionality functions.** Their graphs pass through the point **(0,0)**.

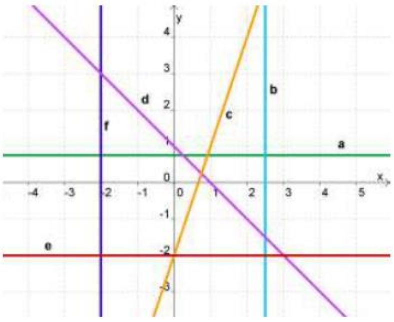
Examples: y=1,60x y=5x

<u>Exercise 15.</u> Find the slope and the ordinate at the origin of the lines:

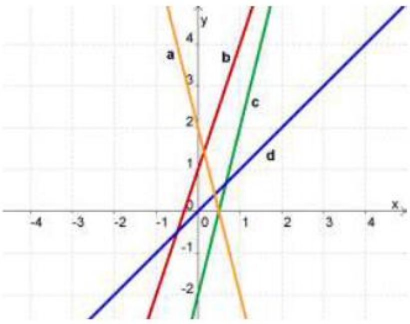
a. y=-5x+2
b. y=-2
c. y=7x
d. 3x+2y=12
e. 2y-4x=5
f. y=1-3x

<u>Exercise 16.</u> Match each line with its equation:

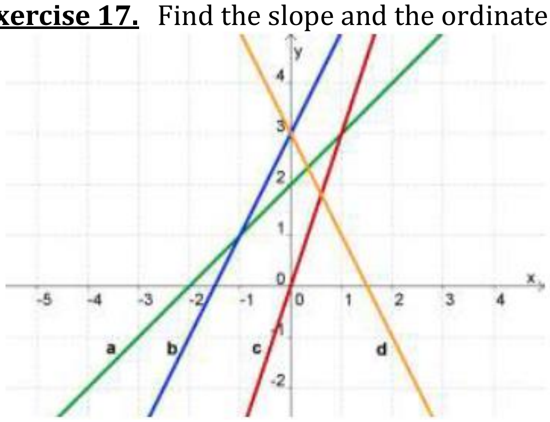
$y=-2$ | $y=-x+1$ | $x=2.5$ | $x=-2$ | $y=0.75$ | $y=3x-2$



$y=4x-2$ | $y=3x+1$ | $y=x$ | $y=2-4x$



<u>Exercise 17.</u> Find the slope and the ordinate at the origin of these graphs. Write the equation of each line:



d. Finding the equation of a straight line

i. If you know the slope and the ordinate at the origin

If you know the slope of a line and the ordinate at the origin you can write the equation of the line.

<u>Example</u>: What is the equation of a line with slope 9 passing through (0,5)?

ii. Point-slope equation

If you know the slope **(m)** and a point **P(x_1, y_1),** on the line, you can find the equation of the line with the following formula:

$$y - y_1 = m(x - x_1)$$

<u>Example</u>: What is the equation of a line with slope 8 that passes through the point (2,7)?

iii. Finding the equation using two points P(x_1, y_1), Q(x_2, y_2)

1. Calculate the slope of the line with the following formula:

If you're given two points

(x_1, y_1) and (x_2, y_2)

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

2. Use the **point-slope equation**.

<u>Example</u>: Find the equation of the line joining (-1,2) and (4,-3)

➔ Since the slope of a straight line tells you how steep it is, **parallel lines** will have the same slope.

<u>Example</u>: Find the equation of a line parallel to y-2x=3.

<u>Exercise 18.</u> Find the equations of the nine lines described in the table:

a) Slope of 7 and intercepts y-axis at (0,5)	**b)** Slope of 0.5 and passes through (0, 3)	**c)** Parallel to a line with slope 4 and passing through (3, 8)
d) Slope of 3 and passing through (4, 7)	**e)** Slope of -2 and cutting through (4, -3)	**f)** Parallel to $y = \frac{1}{4}x - 1$ and passing through (0, -2)
g) Passing through (0, 1) and (1, 5)	**h)** Passing through (0, 2) and (5, 7)	**i)** Passing through the midpoint of (1, 7) and (3, 13) with a slope of 8

4. Studying two linear functions together

a. Relative positions of two linear functions

b. Problems

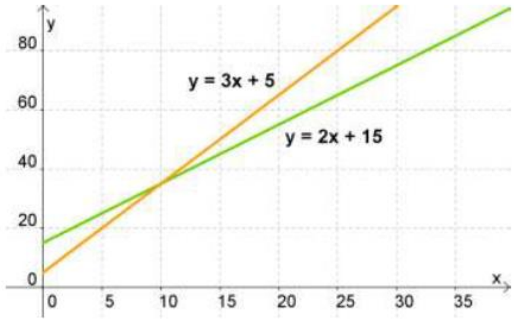
II) *Two competing electricity companies use these formulae to work out customers' bills.*

POWER UP! $y = 3x + 5$

SPARKS ARE US! $y = 2x + 15$

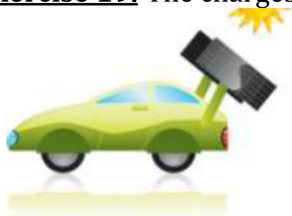
The number of units of electricity used is x. The price of the electricity is £y.

Using graphs, compare the pricing policies of the two companies and advise householders from which company they should buy their electricity.



The graphs show that, from 0 to 10 units of electricity used, the company POWER UP! is cheaper than the company SPARKS ARE US!. However, if the units used are more than 10, the cheapest company is SPARKS ARE US!.

Exercise 19. The charges of two car-hire companies are:



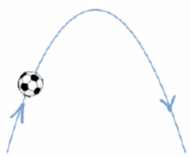
Company 1: 50 € hire fee and 0.2 € per covered kilometre.

Company 2: 20 € hire fee and 0.3 € per covered kilometre.

a) Express the charge of the companies as functions of the covered kilometres.
b) Draw both graphs in the same x and y-axes. Use them to find which company is the most advantageous for the costumers.

5. Quadratic functions $y = ax^2 + bx + c$

If you kick a soccer ball (or shoot an arrow, fire a missile or throw a stone) it will arc up into the air and come down again following the path of a **parabola**.



A function whose graph is a parabola is called a **quadratic function**.

The equation of a **quadratic function** is

$$y = ax^2 + bx + c \quad \text{, where } a \neq 0.$$

Examples: $y = x^2$, $y = x^2 - 4$, $y = -3x^2 + 2x + 1$

The domain of a quadratic function is $\mathbb{R}$.

a. Representar funciones cuadráticas - Graphing quadratic functions

i. Orientation

If $a > 0$, the parabola opens upward.
If $a < 0$, the parabola opens downward.

ii. Vertex

Vertex: $V(x_v, y_v)$

$x_v = \dfrac{-b}{2a}$. The axis of symmetry of the parabola is the vertical line $x = x_v$.

iii. Where the parábola intersects with the axes

<u>x-intercepts</u>:

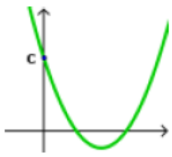
An x-intercept is a point on the graph where $y = 0$.

If $y = 0 \Rightarrow ax^2 + bx + c = 0$. When we solve the equation we can have:

<u>y-intercept</u>:

A y-intercept is a point on the graph where $x = 0$.

If $x = 0 \Rightarrow y = c$. Then the y-intercept point is $(c, 0)$.



iv. Table of values with points on both sides of the vertex

Plot some points whose abscissa is close to the vertex of the parabola (on both sides of it).

v. Graph the parabola

<u>Exercise 20.</u> Graph the following quadratic functions:

- **a.** $y = x^2 + 2x - 3$
- **b.** $y = x^2 - 5$
- **c.** $y = -x^2 + 5x$
- **d.** $y = x^2 - 2x + 1$
- **e.** $y = x^2 - 2x + 5$
- **f.** $y = -x^2 - 1$
- **g.** $y = -x^2 + 10x - 8$

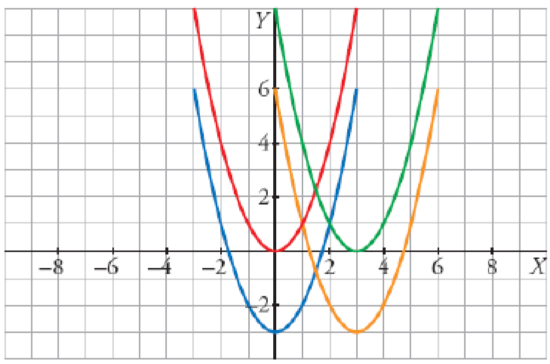
<u>Exercise 21.</u> A ball is thrown into the air. The function $h = 20t - 5t^2$ shows its height "h" metres, above the ground "t" seconds later it is thrown with an initial speed of 20m/s.

- **a.** Graph the function and find its domain.
- **b.** Find the maximum height reached by the ball and the time at which it reaches this height.
- **c.** Find the interval of time when the ball is above 15 metres

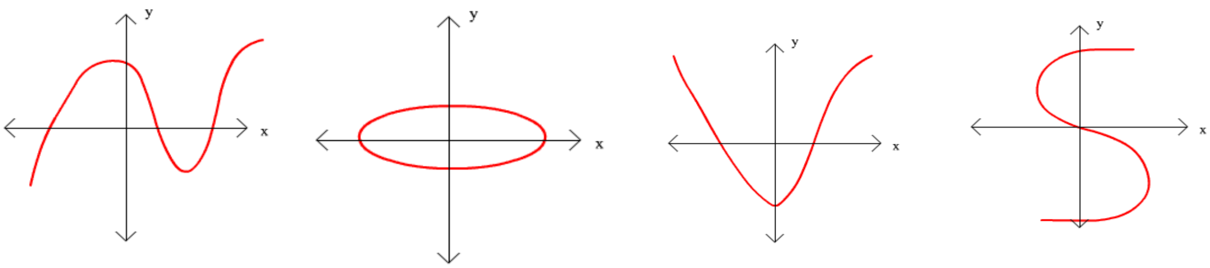
<u>Exercise 22.</u> The annual cost, in euros, of producing "x" computers in a company is $C(x) = 20000 + 250x$. The annual income, in euros, that the company gets by selling "x" computers is $I(x) = 600x - 0,1x^2$. How many computers does the company have to sell each year to maximize the profit?

<u>Exercise 23.</u> Match each parabola with its equation:

a) $y = x^2$ b) $y = x^2 - 3$ c) $y = (x-3)^2$ d) $y = x^2 - 6x + 6$



<u>Exercise 24.</u> Determine whether the graph is a function and explain your reasoning.



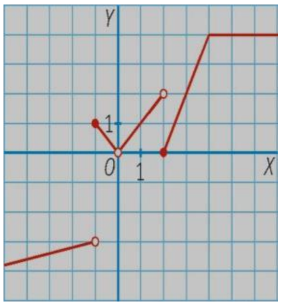
Exercise 25. When taking a taxi, we have to pay a fixed fee of €2.50 and €0.80 per kilometer.

a. State the formula that represents the relationship between the number of kilometers and the final cost.
b. If we go over 5 kilometers, how does it cost?
c. Plot the graph.

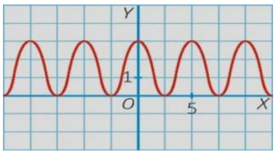
Exercise 26. Represents the following piecewise functions::

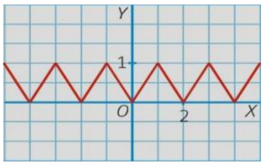
a) $f(x)=\begin{cases} -2 & si \quad x \leq 0 \\ -x+2 & si \quad 0 < x \leq 5 \end{cases}$ b) $f(x)=\begin{cases} 2x-2 & si \quad -3 \leq x \leq 1 \\ -x+1 & si \quad 1 < x \leq 2 \end{cases}$ c) $f(x)=\begin{cases} 3x-4 & si \quad x < 1 \\ 0 & si \quad 1 < x < 3 \\ -2x+5 & si \quad x \geq 3 \end{cases}$

Exercise 27. Determine the **domain**, **range**, and **continuity** of the function. Also find f (1), f (2), and f (3). Is there f (-1)?

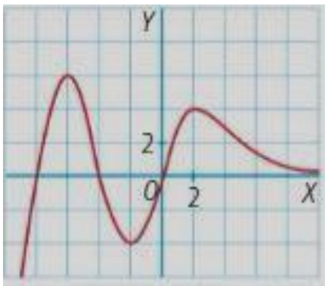


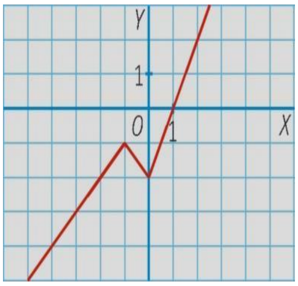
Exercise 28. Determine the **period** of the following graphing functions. Find f (100), f (1401), f (2017) in both functions.





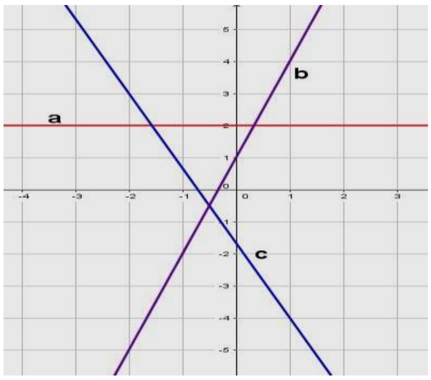
Exercise 29. Study the monotony of the following functions and determine the relative and absolute extremes:





Exercise 30. Determine the equation of the line:

a. That passes through points A (1,3) and B (2,5). Later indicate the slope.
b. Which has slope -2 and passes through the origin of coordinates
c. Its ordinate at the origin is 3 and it passes through the point (-4,3). Later indicate the slope.
d. It has slope -1 and goes through the point (4,0)

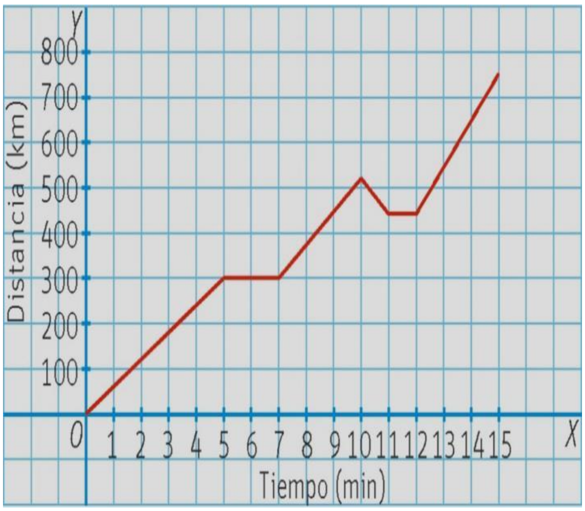


Exercise 31. Find the equation of each line represented, previously determining the respective slope:

Exercise 32. A bus company A charges € 1.75 for management costs and 7 cents for each kilometer travelled. Another company B charges € 1 for management costs and 9 cents per kilometer.

a. Identify the variables and obtain the cost expression based on the km travelled for each bus company.
b. Graph the price in cents of a bus ticket based on the distance in kilometers that separates the origin and destination. Use a suitable scale for the two variables.
c. Calculate the distance for which the two companies charge the same for the ticket. What price is that?
d. When is one more profitable than the other?
e. If the distance between two cities is 200 km, which company is interested in buying the ticket?

Exercise 33. María sale de casa para ir a clase a las 8 de la mañana. Va a una velocidad constante al principio, pero se encuentra con una amiga y se para con ella. Después, aumenta su velocidad porque, si no, no llega a tiempo. Cuando está cerca del instituto, se da cuenta de que tiene que comprar un bolígrafo y se ha pasado la papelería, vuelve a la papelería y cuando tiene el bolígrafo, se va rápidamente a clase. La siguiente gráfica representa la distancia (en metros) de María a su casa, en función del tiempo (en minutos):



Contesta a las siguientes preguntas:

a. ¿Cuánto tiempo ha tardado María en llegar a clase?

b. ¿Cuánto tiempo está parada con la amiga?, ¿a qué hora se encuentra con ella?

c. ¿Cuánto tiempo tarda en volver a la papelería?

d. ¿Cuánto tiempo transcurre desde que deja a la amiga hasta que llega a clase?

e. ¿A qué distancia de su casa se encuentra con la amiga?

f. ¿Qué distancia hay de su casa al centro?

g. ¿El camino es cuesta arriba o cuesta abajo?

Exercise 34. Find the equation of the lines that pass through the points:

a) A(4,-1) y B(-2,4) b) A(2,0) y B(-3,1) c) A(-1/4, -1) y B(2, -3)

Exercise 35. The GUAYANDÚ internet server has the GUAY rate which consists of a fixed monthly fee of € 20 and € 0.01 per minute. The JOMEIL internet server has the CHUPY rate with no fixed fee. In this mode, you only have to pay € 0.02 per minute.

- **a.** Identify the variables and obtain the cost expression in terms of minutes for each rate. What kind of function have you obtained?
- **b.** Represents both usage time-expense functions on the same axes.
- **c.** How many minutes will we pay the same with both rates? How much would be paid?
- **d.** From how many minutes per month is GUAY more profitable than CHUPY?
- **e.** If I usually surf 1500 minutes per month, what rate is best for me?

Exercise 36. Jaime needs a car to rent during his vacation. You have contacted two car rental agencies and the offers have been:

Agency A: 10 euros for the rental plus 5 cents per km travelled with the car.

Agency B: 5 euros for the rental plus 8 cents per km travelled with the car.

- a. Identify the variables and obtain the cost expression based on the km travelled for each agency. What kind of function have you obtained?
- b. Represents on the same axes both functions kms travelled-cost.
- c. From how many kms is one more profitable than the other?
- d. If Jaime will travel 230 km, which agency is best for him? How much will he pay in total for those 230 km when choosing the most profitable agency?

Exercise 37. Draw these functions:

a) $f(x)=x^2-2x-3$ b) $f(x)= -2x^2+4x+6$ c) $f(x)= x^2-4$ d) $f(x)= x^2-6x+11$

e) $f(x)= x^2-4x$ f) $f(x)= x^2-6x+10$ g) $f(x)= -2x^2+8x-8$ h) $f(x)= x^2+9$

Exercise 38. An arrow is launched whose height, h, with respect to time, t, is given by the equation $h = -5t^2 + 60t$ (h in meters and t in seconds).

a. Represent the function graph on these axes.

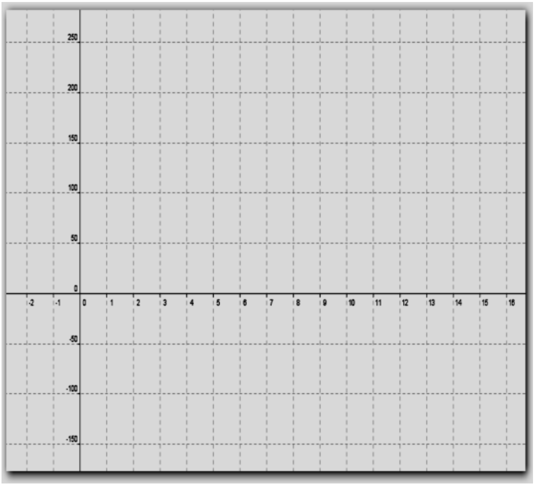
b. How long after the arrow is released does it reach its maximum height?

c. What is that maximum height?

d. How many seconds does the arrow fall to the ground?

e. How high was the arrow at 3 seconds?

f. How many seconds later was the arrow 100 meters high?



Exercise 39. In the following functions, study domain, range, monotony and relative and absolute extremes.